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Spinors and related tensors invariant under E(3) and its subgroups

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Abstract. Necessary and sufficient conditions of invariance on (first- and second-rank) spinors under the Poincaré group are established through an infinitesimal method. The group E(3) and its subgroups are then specifically considered and their invariant first-rank spinors are derived. Second-rank spinors and four-vectors are also connected through invariance arguments, and their consideration enlightens some geometrical concepts such as Lie derivatives of spinor fields.

1. Introduction

Tensor fields invariant under some maximal subgroups of the conformal group of space–time have already been derived in this journal (Beckers *et al* 1979). Let us hereafter denote this reference by I. It belongs to a series of recent studies on the Poincaré and (or) conformal groups and their subgroups (Bacry *et al* 1970, Combe and Sorba 1975, Beckers and Comté 1976, Beckers *et al* 1978, Beckers and Hussin, to be published) and their implications on invariant tensor fields.

Solutions to the Yang–Mills equations invariant under compact subgroups of the conformal group have also been studied (Harnad and Vinet 1978, Harnad *et al* 1979a), and further interesting results were obtained (Harnad *et al* 1979b) in connecting group actions on principal bundles and invariance conditions for gauge fields.

More recently, invariant *spinor* fields have been systematically derived when Poincaré or conformal symmetries are under consideration (Beckers *et al* 1980). In fact, non-trivial results were obtained in connection with the problem of determining the spinor fields which are invariant under subgroups of dimension $\leqslant 6$ of the Poincaré group P(3, 1).

Let us note that such results already permit the construction of interaction terms in Lagrangians (analogously to the quantum electrodynamical case $A_\mu \bar{\Psi}\gamma^\mu\Psi$): these terms can effectively be made up of four-vectors and Dirac spinors invariant under a specific Poincaré subgroup. Furthermore, in connection with generalised Yang–Mills fields and equations, such methods could also be of special interest in gauge theories with specific symmetries.

Here we want to take into account the specific transformation laws of *spinors*, first under the Poincaré group and then under the Euclidean group in three dimensions. Through an infinitesimal method analogous to the one developed in I, we then impose the invariance of these spinors in order to obtain necessary and sufficient conditions

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corresponding to those already obtained (in I for example) on tensor fields in general and on four-vectors in particular. Moreover, besides specific results obtained through the example of the group E(3) and its subgroups, we can learn here how Lie derivatives act on spinors, a relatively elaborate notion of differential geometry (Kosmann 1966, 1972, and references therein, Jhangiani 1978). In fact, this article is composed as follows: § 2 is devoted to the notation according to I and to a few generalities on the Poincaré group, on E(3) and its subgroups, and on spinors. Section 3 contains the discussion of invariance conditions on spinors with respect to the Poincaré group and its subgroups: the relation between second-rank, mixed, hermitian spinors and four-vectors is the key of these developments and the invariance conditions on first-rank (or fundamental) spinors are explicitly established. The case of the subgroup E(3) and its own subgroups is considered in § 4: the invariant fundamental spinors are explicitly derived. Finally, in § 5, returning to second-rank, mixed, hermitian spinors and to four-vectors, we express their invariance through Lie derivatives, showing how such concepts are working on spinors. We also discuss consistencies between invariance considerations and spin–tensor correspondences, by taking into account other recent results on invariant four-vectors (Beckers and Hussin, to be published) when E(3) and some of its subgroups are under study.

2. Notation and generalities on E(3) and spinors

According to the notation used in I, we deal with the space–time events $x \equiv \{x^\mu, \mu = 0, 1, 2, 3\} \equiv \{t, \boldsymbol{r}\}$ and the Minkowski metric tensor $G_M \equiv \{g^{\mu\nu}\} = \mathrm{diag}\,(1, -1, -1, -1)$. We also consider the *infinitesimal* forms of the *continuous* coordinate transformations

$$x \overset{\alpha}{\to} x' \colon x'^\mu = x^\mu + \alpha^\mu, \tag{2.1}$$

$$x \overset{\omega}{\to} x' \colon x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu, \qquad \omega^{\mu\nu} = -\omega^{\nu\mu}, \tag{2.2}$$

corresponding to space–time translations and to restricted Lorentz rotations respectively. These elements lead to the well known restricted inhomogeneous Lorentz or Poincaré group E(3, 1), a (non-maximal) subgroup of the conformal group of space–time, already discussed in I. The Lie algebra associated with equations (2.1) and (2.2) is ten-dimensional and is generated by $\{P^\mu, M^{\mu\nu} \equiv (\boldsymbol{J}, \boldsymbol{K})\}$. The $\boldsymbol{J}$'s refer to spatial rotations and the $\boldsymbol{K}$'s to pure Lorentz transformations. These transformations can be parametrised in a very convenient way by

$$\theta^i = \tfrac{1}{2}\epsilon^{ijk}\omega_{jk} \qquad (\epsilon^{123} = 1), \qquad \phi^i = \omega^{0i}, \qquad i, j, k = 1, 2, 3, \tag{2.3}$$

so that

$$\{\omega^{\mu\nu}\} \equiv (\boldsymbol{\phi}, \boldsymbol{\theta}).$$

From I, and with such a parametrisation, it is easy to obtain the necessary and sufficient conditions of invariance of a general four-vector denoted $A \equiv (A^\mu) = (V, \boldsymbol{A})$ under infinitesimal Poincaré transformations corresponding (from (2.1) and (2.2)) to the form

$$x \xrightarrow{(\alpha,\, \omega)} x' \colon x'^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu + \alpha^\mu \tag{2.4}$$

$$= x^\mu + \xi^\mu. \tag{2.5}$$

This has already been obtained in § 4 of I by using an infinitesimal method applied to *point* transformations. From equations (4.14) in I, we immediately obtain

$$DA(x)+\omega\,.\,A(x)=0 \tag{2.6}$$

where D is a differential operator given by

$$D\equiv x\,.\,\omega\,.\,\nabla-a\,.\,\nabla=x_\rho\omega^{\rho\lambda}\partial_\lambda-\alpha^\lambda\partial_\lambda \tag{2.7}$$

$$\equiv(\boldsymbol{r}\,.\,\boldsymbol{\phi})\partial/\partial t+(t\boldsymbol{\phi}+\boldsymbol{r}\Lambda\boldsymbol{\theta})\,.\,\partial/\partial\boldsymbol{r}-\alpha\,.\,\nabla. \tag{2.8}$$

Through (2.3), we have

$$\omega\,.\,A\equiv(-\boldsymbol{\phi}\,.\,\boldsymbol{A},\,-V\boldsymbol{\phi}+\boldsymbol{\theta}\Lambda\boldsymbol{A}) \tag{2.9}$$

so that the necessary and sufficient conditions (2.6) become

$$DV(x)-\boldsymbol{\phi}\,.\,\boldsymbol{A}(x)=0 \tag{2.10}$$

and

$$D\boldsymbol{A}(x)+\boldsymbol{\theta}\Lambda\boldsymbol{A}(x)-V(x)\boldsymbol{\phi}=0. \tag{2.11}$$

These conditions correspond to equations (4.22) in I, restricted to the Poincaré context. As a final remark, we notice that these conditions are equivalent to the expression of vanishing Lie derivatives (L_X) of one-forms A, with respect to the vector fields X defined by equation (2.5):

$$X\equiv\xi^\mu\partial_\mu. \tag{2.12}$$

This can be written

$$L_XA=0 \qquad \text{or} \qquad \xi^\alpha\partial_\alpha A_\mu+A_\alpha\partial_\mu\xi^\alpha=0 \tag{2.13}$$

where A is a *covariant* four-vector. Other information on this geometrical approach can be found in § 6 of I and in the work of Beckers *et al* (1978).

Let us now mention that the subgroup structure of the Poincaré group has already been pointed out by different authors (Bacry *et al* 1974, Patera *et al* 1974). It evidently shows that the Euclidean group in three dimensions, E(3), belongs to such a structure: E(3) is a subgroup of dimension six, which by definition is the group of transformations of the three-dimensional (real) vector space leaving the Euclidean distance between two points invariant. Its Lie algebra ($\boldsymbol{J}$, $\boldsymbol{P}$) is generated by three operators $\boldsymbol{P}$ (associated with spatial translations) and by three $\boldsymbol{J}$'s (associated with true rotations) satisfying the following commutation relations:

$$[J^i,J^k]=\mathrm{i}\,\epsilon^{ikl}J^l, \qquad [J^i,P^k]=\mathrm{i}\,\epsilon^{ikl}P^l, \qquad [P^i,P^k]=0. \tag{2.14}$$

E(3) is also the symmetry group of the free, time-independent, Schrödinger equation of non-relativistic quantum mechanics. Its own subgroup structure—also well known (Bacry *et al* 1974)—has already been exploited in connection with symmetry breaking considerations (Beckers *et al* 1977) and with invariant *tensor* fields (Beckers and Comté, to be published, Beckers and Hussin, to be published). Let us recall briefly this E(3)-subgroup structure: there are *ten* non-trivial and non-equivalent subgroups of

$E(3) \equiv \{\boldsymbol{J}, \boldsymbol{P}\}$:

one of dimension $n=4$: $E(2)\otimes T(1) \equiv (J^3, \boldsymbol{P}\}$;

four of dimension $n=3$: $T(3) \equiv \{\boldsymbol{P}\}$, $O(3) \equiv \{\boldsymbol{J}\}$, $E(2) \equiv \{J^3, P^1, P^2\}$, $\overline{E(2)} \equiv \{J^3 + mP^3, P^1, P^2, m \neq 0\}$;

two of dimension $n=2$: $T(2) \equiv \{P^1, P^2\}$, $O(2)\otimes T(1) \equiv \{J^3, P^3\}$; (2.15)

three of dimension $n=1$: $T(1) \equiv \{P^3\}$, $O(2) \equiv \{J^3\}$, $\overline{D(2)} \equiv \{J^3 + mP^3, m \neq 0\}$.

The infinitesimal E(3)-transformations are particular cases of equations (2.4) and (2.5). They are

$$x \xrightarrow{(\alpha^i, \theta^i)} x' : x'^i = x^i + \alpha^i + \omega^i{}_k x^k \tag{2.16}$$

$$= x^i + \xi'^i. \tag{2.17}$$

For example, we can immediately obtain the conditions corresponding to equations (2.8), (2.10) and (2.11). Indeed, we obtain

$$D' V(\boldsymbol{x}) = 0, \qquad D'\boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\theta} \Lambda \boldsymbol{A}(\boldsymbol{x}) = 0, \tag{2.18}$$

with

$$D' \equiv (\boldsymbol{r} \Lambda \boldsymbol{\theta} + \boldsymbol{\alpha}) \,.\, \partial/\partial \boldsymbol{r}. \tag{2.19}$$

Such necessary and sufficient conditions have already been exploited in studying invariant four-potentials (Beckers and Hussin, to be published).

In the following sections we want to undertake a corresponding study for spinors. These are well known entities (Bade and Jehle 1953, Pirani 1964, Parke and Jehle 1965, and references therein, Misner *et al* 1973). The transformation laws of first-rank (or fundamental) contravariant and covariant spinors (labelled with undotted indices) and of their complex conjugate spinors (labelled with dotted indices) are

$$\xi'^A = L^A{}_B \xi^B, \qquad \eta'_A = \eta_B (L^{-1})^B_A, \tag{2.20}$$

$$\xi'^{\dot U} = L^{*\dot U}{}_{\dot V} \xi^{\dot V}, \qquad \eta'_{\dot U} = \eta_{\dot V} (L^{*-1})^{\dot V}_{\dot U}, \tag{2.21}$$

where $L \in SL(2, C)$, L^* is the complex conjugate matrix associated with L, and $A, B, \ldots, U, V, \ldots = 1, 2$. In such a context, raising or lowering indices is easily realised through the metric spinor,

$$C \equiv (C^{AB}) = (C_{AB}) = (C^{\dot U \dot V}) = (C_{\dot U \dot V}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.22}$$

and spinor analysis tells us that spinors of higher rank may be formed of spinors of lower rank (by multiplication) and conversely (by contraction), etc.

We also know that there exist meaningful correspondences between spinors and tensors. The simplest one is the correspondence between a mixed '1,1-spinor' and a '1-index tensor' or four-vector (see for example Misner *et al* 1973, p 1151): with each *real* four-vector $A \equiv (A^\mu)$ is associated a mixed, second-rank, *hermitian* spinor, so that we can write

$$\chi_{\dot U A} = \sigma^\mu{}_{\dot U A} A_\mu \tag{2.23}$$

where $\sigma^\mu{}_{\dot U A}$ are the Infeld–Van der Waerden symbols (Infeld and Van der Waerden 1933), whose properties are very well known. In particular, such symbols do transform as four-vectors with respect to Lorentz transformations ($\mu = 0, 1, 2, 3$) and as second-rank spinors with respect to spinor transformations ($A, U = 1, 2$). Then their transformation properties can be written

$$\sigma'^{\mu\dot{U}A} = \Lambda^{\mu}{}_{\nu} L^{*\dot{U}}{}_{\dot{V}} L^{A}{}_{B} \sigma^{\nu\dot{V}B} \tag{2.24}$$

where Λ is the Lorentz transformation connecting the primed and unprimed systems of reference, and has the infinitesimal form $\Lambda \sim 1 + \omega$.

3. Invariant spinors and the Poincaré group

The main results of this section have already been obtained (Beckers *et al* 1980), but we want to apply the infinitesimal method through spin–tensor correspondences and invariance arguments on tensors and on spinors separately. In fact, let us exploit the relation (2.23) connecting second-rank, mixed, hermitian spinors with four-vectors. If we remember that four-vectors invariant under the Poincaré group have already been derived (Beckers and Hussin, to be published), the corresponding invariant second-rank spinors can then be simply determined from these equations (2.23). We only have to impose invariance of the Infeld–van der Waerden symbols, i.e. to express, following their transformation properties (2.24), that

$$\sigma^{\mu\dot{U}A} = \Lambda^{\mu}{}_{\nu} L^{*\dot{U}}{}_{\dot{V}} L^{A}{}_{B} \sigma^{\nu\dot{V}B}. \tag{3.1}$$

From these conditions, it is possible to determine the explicit form of the matrix L when infinitesimal Lorentz transformations $\Lambda \sim 1 + \omega$ are considered. A rather lengthy but simple calculation leads to

$$L^{A}{}_{B} = \delta^{A}{}_{B} + \tfrac{1}{2}\omega_{\mu\nu}\sigma^{\mu\dot{U}A}\sigma^{\nu}{}_{\dot{U}B}. \tag{3.2}$$

Such a relation can be written in a compact form if we notice that the Infeld–Van der Waerden symbols are usually chosen in the following form (Pirani 1964):

$$(\sigma^{\mu}{}_{\dot{U}A}) = 2^{-1/2}(\mathbb{1}, \boldsymbol{\sigma}), \qquad (\sigma^{\mu\dot{U}A}) = 2^{-1/2}(\mathbb{1}, -\boldsymbol{\sigma}^{*}), \tag{3.3}$$

where the $\boldsymbol{\sigma}$'s are the Pauli matrices. We obtain, from equations (3.2) and (3.3),

$$L = \mathbb{1} - \tfrac{1}{2}(\boldsymbol{\Omega} \cdot \boldsymbol{\sigma}) \tag{3.4}$$

with

$$\boldsymbol{\Omega} \equiv \{\Omega^{k} = \phi^{k} + \mathrm{i}\,\theta^{k}, k = 1, 2, 3\}. \tag{3.5}$$

Let us insist on the fact that equation (3.4) corresponds to a specific *invariance* of *spinorial* quantities.

Now, as already mentioned, spinor analysis tells us in particular that an arbitrary-rank spinor may be formed of spinors of lower rank. Then the *fundamental* spinors associated with the fundamental representations of SL(2, C) are of special interest. Let us establish the necessary and sufficient conditions of invariance of a fundamental spinor transforming according to equation (2.20). Under infinitesimal Poincaré transformations, the invariance is expressed by

$$\xi'^{A}(x') = \xi^{A}[(1+\omega)x + \alpha] = L^{A}{}_{B}\xi^{B}(x) \tag{3.6}$$

or

$$\xi^{A}(x) + (\omega \cdot x + \alpha)^{\rho}\partial_{\rho}\xi^{A}(x) = \xi^{A}(x) - \tfrac{1}{2}(\boldsymbol{\Omega} \cdot \boldsymbol{\sigma})^{A}{}_{B}\xi^{B}(x).$$

We finally obtain

$$D\xi^A(x)=\tfrac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma})^A{}_B\xi^B(x) \qquad (A=1,2) \tag{3.7}$$

where D is given by equation (2.8). Explicitly, these invariance conditions are

$$\Omega^3\xi^1(x)+(\Omega^1-\mathrm{i}\Omega^2)\xi^2(x)=2D\xi^1(x) \tag{3.8a}$$

and

$$(\Omega^1+\mathrm{i}\Omega^2)\xi^1(x)-\Omega^3\xi^2(x)=2D\xi^2(x). \tag{3.8b}$$

This result is equivalent to that obtained elsewhere (Beckers *et al* 1980) when $(\frac{1}{2},0)$-spinors are considered. Moreover, by using equations (3.8), it is also very easy to show that there exists a 'kinematical' group (Bacry *et al* 1970) of a fundamental *uniform and constant* spinor. This 'kinematical' group is of dimension six; it is generated by the four P^μ's and the two generators of A(2):

$$A^1=J^2-K^1, \qquad A^2=J^1+K^2. \tag{3.9}$$

In order to be complete (in connection with other contributions: Combe and Sorba 1975, Beckers and Hussin, to be published), we can also define the *stabiliser* of the fundamental spinor $\xi(x)$ in the Poincaré group P as:

$$\mathrm{Stab_P}(\xi)=\{(a,\Lambda)\colon (a,\Lambda)\in \mathrm{P} \quad \text{and} \quad {}^{(a,\Lambda)}\xi(x)=\xi(x)\}. \tag{3.10}$$

Then we easily obtain the following result: if a subgroup G of P stabilises the spinor ξ, then at any point x_0 belonging to Minkowski space for which $\xi(x_0)$ has a non-zero finite value, the set of transformations in G of the form $(x_0-\Lambda x_0, \Lambda)$ generates a Lie algebra of dimension $\leqslant 2$, conjugate to the homogeneous part of the Lie algebra of the 'kinematical' group.

In combining these results, we deduce that the 'kinematical' group $\{P^\mu, A^1, A^2\}$ is the largest Poincaré subgroup which admits a non-zero fundamental spinor, and that the only Poincaré subgroups which are stabilisers of ξ in P are among those of dimension $\leqslant 6$. These results lead to the discussion of SPIPS and NONSPIPS as discussed elsewhere (Beckers *et al* 1980).

Here let us apply the method—and more particularly the conditions (3.8)—to the case of the group E(3) and its subgroups (2.15), all cases characterised by dimensions less than or equal to six, the dimension of the 'kinematical' group.

4. Invariant fundamental spinors under E(3) and its subgroups

When E(3)-transformations of the form (2.16) are considered, the necessary and sufficient conditions of invariance of a fundamental spinor (3.8) become

$$\begin{pmatrix} \mathrm{i}\theta^3-2D' & \mathrm{i}\theta^1+\theta^2 \\ \mathrm{i}\theta^1-\theta^2 & -\mathrm{i}\theta^3-2D' \end{pmatrix}\begin{pmatrix}\xi^1(x)\\ \xi^2(x)\end{pmatrix}=0 \tag{4.1}$$

with

$$D'\equiv(\boldsymbol{r}\wedge\boldsymbol{\theta}+\boldsymbol{\alpha})\cdot\partial/\partial\boldsymbol{r}\equiv(2.19). \tag{4.2}$$

Let us exploit these conditions in connection with the cases of E(3) and its ten non-trivial subgroups (2.15). Firstly, we notice that only two of the eleven algebras contain an homogeneous part of dimension greater than that of the algebra of A(2):

these are the E(3) and O(3) cases. Then we immediately conclude that there are no non-trivial invariant spinors in these two cases. Such a result can also be obtained by dealing directly with equations (4.1) and (4.2). Secondly, let us apply this method to the (nine) other cases: we shall find two more trivial results and seven non-trivial spinors.

Invariance under P^i $(i = 1, 2, 3)$ corresponds to the parameters

$$\boldsymbol{\alpha}^{(i)} \equiv (\delta_1^i, \delta_2^i, \delta_3^i), \qquad \boldsymbol{\theta} \equiv 0, \tag{4.3}$$

and to the differential operator (4.2)

$$D'^{(i)} \equiv -\partial/\partial x^i. \tag{4.4}$$

Under P^1, P^2 or P^3 we obtain respectively

$$\xi^1 = \xi^1(y, z), \qquad \xi^2 = \xi^2(y, z), \tag{4.5}$$

$$\xi^1 = \xi^1(x, z), \qquad \xi^2 = \xi^2(x, z), \tag{4.6}$$

or

$$\xi^1 = \xi^1(x, y), \qquad \xi^2 = \xi^2(x, y). \tag{4.7}$$

Under *all* the $\boldsymbol{P}$'s, we evidently deduce the *constant* character of the two spinor components.

Invariance under J^3 corresponds to the parameters

$$\boldsymbol{\theta}^{(3)} \equiv (0, 0, \delta_3^3) = (0, 0, 1), \qquad \boldsymbol{\alpha} \equiv 0, \tag{4.8}$$

and to the differential operator (4.2)

$$D''^{(3)} \equiv y\partial/\partial x - x\partial/\partial y. \tag{4.9}$$

Now the different cases can be discussed in a very simple way.

Invariance under E(2)⊗T(1), $n = 4$, evidently gives a trivial result: $\xi \equiv 0$.

Invariance under T(3), $n = 3$, requires the constant character of the two components:

$$\xi^1 = C, \qquad \xi^2 = C'. \tag{4.10}$$

Invariance under E(2), $n = 3$, leads to a trivial result although invariance under $\overline{\mathrm{E}(2)}$, $n = 3$, will give us a non-trivial spinor. Indeed, the P^1 and P^2 generators impose only a z-dependence of the components ξ^1 and ξ^2 and the system (4.1) becomes, in the $\overline{\mathrm{E}(2)}$ case, equivalent to the equations $(m \neq 0)$

$$\mathrm{i}\xi^1(z) + 2m\partial\xi^1(z)/\partial z = 0, \tag{4.11a}$$

$$\mathrm{i}\xi^2(z) - 2m\partial\xi^2(z)/\partial z = 0, \tag{4.11b}$$

so that we finally obtain

$$\xi^1(z) = C\,\mathrm{e}^{-\mathrm{i}z/2m}, \qquad \xi^2(z) = C'\,\mathrm{e}^{\mathrm{i}z/2m} \tag{4.12}$$

where C and C' are two constants.

Invariance under T(2), $n = 2$, evidently gives the spinor

$$\xi^1 = \xi^1(z), \qquad \xi^2 = \xi^2(z). \tag{4.13}$$

Invariance under O(2)⊗T(1), $n = 2$, will be simple to express if cylindrical coordinates are chosen:

$$\rho = (x^2 + y^2)^{1/2}, \qquad \phi = \tan^{-1}(y/x), \qquad z = z, \tag{4.14}$$

because the differential operator (4.9) is in this case

$$D''^{(3)} \equiv -\partial/\partial\phi \tag{4.15}$$

and the system (4.1) reduces to

$$(\mathrm{i}+2\partial/\partial\phi)\xi^1(\rho,\phi)=0, \tag{4.16a}$$

$$(\mathrm{i}-2\partial/\partial\phi)\xi^2(\rho,\phi)=0, \tag{4.16b}$$

when P^3 invariance is also taken into account. The invariant spinor reads

$$\xi^1(\rho,\phi)=C(\rho)\,\mathrm{e}^{-\mathrm{i}\phi/2}, \qquad \xi^2=C'(\rho)\,\mathrm{e}^{\mathrm{i}\phi/2}. \tag{4.17}$$

Invariance under T(1) or O(2), $n=1$, gives the respective results

$$\xi^1=\xi^1(x,y), \qquad \xi^2=\xi^2(x,y), \tag{4.18}$$

or

$$\xi^1=C(r,\theta)\,\mathrm{e}^{-\mathrm{i}\phi/2}, \qquad \xi^2=C'(r,\theta)\,\mathrm{e}^{\mathrm{i}\phi/2}, \tag{4.19}$$

when, in the second case, polar coordinates (r,θ,ϕ) are used.

Finally, invariance under $\overline{\mathrm{O}(2)}$, $n=1$, becomes simple if we introduce helical coordinates, i.e.

$$\rho=(x^2+y^2)^{1/2}, \qquad u=\tfrac{1}{2}(\phi+z/m), \qquad v=\tfrac{1}{2}(z/m-\phi). \tag{4.20}$$

The corresponding differential operator (4.2) then takes the form

$$D\equiv -\partial/\partial u \tag{4.21}$$

and we obtain the invariant spinor:

$$\xi^1=C(\rho,z/m-\phi)\exp[-\tfrac{1}{4}\mathrm{i}\,(\phi+z/m)], \qquad \xi^2=C'(\rho,z/m-\phi)\exp[\tfrac{1}{4}\mathrm{i}\,(\phi+z/m)]. \tag{4.22}$$

5. Invariant second-rank spinors and four-vectors

Let us consider again second-rank spinors and four-vectors and their invariance properties under the Poincaré group. The infinitesimal method directly applied to second-rank spinors leads us to

$$\chi'^{\dot U A}(x')=\chi^{\dot U A}[(1+\omega)x+\alpha]=L^{*\dot U}{}_{\dot V}L^A{}_B\chi^{\dot V B}(x) \tag{5.1}$$

where we have to use the infinitesimal form (3.4) of the matrix L. Explicit developments of equations (5.1) give the invariance conditions in the form

$$(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma})^{*\dot U}{}_{\dot V}\chi^{\dot V A}(x)+(\boldsymbol{\Omega}\cdot\boldsymbol{\sigma})^A{}_B\chi^{\dot U B}=2D\chi^{\dot U A}(x). \tag{5.2}$$

Written explicitly in terms of the *covariant* components of the spinor, these conditions become

$$\begin{aligned}
&(\Omega^1-\mathrm{i}\Omega^2)\chi_{\dot 21}+(\Omega^1-\mathrm{i}\Omega^2)^*\chi_{\dot 12}-2(\phi^3-D)\chi_{\dot 22}=0,\\
&(\Omega^1+\mathrm{i}\Omega^2)\chi_{\dot 22}+(\Omega^1-\mathrm{i}\Omega^2)^*\chi_{\dot 11}+2(\mathrm{i}\theta^3+D)\chi_{\dot 21}=0,\\
&(\Omega^1-\mathrm{i}\Omega^2)\chi_{\dot 11}+(\Omega^1+\mathrm{i}\Omega^2)^*\chi_{\dot 22}-2(\mathrm{i}\theta^3-D)\chi_{\dot 12}=0,\\
&(\Omega^1+\mathrm{i}\Omega^2)\chi_{\dot 12}+(\Omega^1+\mathrm{i}\Omega^2)^*\chi_{\dot 21}+2(\phi^3+D)\chi_{\dot 11}=0.
\end{aligned} \tag{5.3}$$

Let us show that such a system can easily be recovered from vanishing Lie derivatives on spinors. In § 2 we recalled such a geometrical concept on four-vectors (cf equations (2.12) and (2.13)) and this is very often used when *differential forms* are considered. If *spinor fields* are concerned, the concept of Lie derivative is usually not so clear, and a specific literature (Kosmann 1966, 1972 and references therein, Jhangiani 1978) has already dealt with such a subject. Here we can express this invariance by

$$L_X\chi_{\dot{U}A}=0, \qquad \forall U, A=1,2, \tag{5.4}$$

where the vector fields X are once again defined by (2.12). Now, if we consider *hermitian* spinors we can use (2.23) and obtain

$$L_X\chi_{\dot{U}A}=L_X(\sigma^{\mu}{}_{\dot{U}A}A_{\mu})=\sigma^{\mu}{}_{\dot{U}A}(L_XA_{\mu}) \tag{5.5}$$

because the Infeld–Van der Waerden symbols are constants and L_X is a linear operator. Consequently, in connection with equation (5.4), we obtain

$$L_X\chi_{\dot{U}A}=0 \leftrightarrow L_XA_{\mu}=0 \tag{5.6}$$

and equations (5.4) and (5.5) give

$$L_X\chi=\frac{1}{\sqrt{2}}\begin{pmatrix} L_X(A_0+A_3) & L_X(A_1-\mathrm{i}A_2) \\ L_X(A_1+\mathrm{i}A_2) & L_X(A_0-A_3)\end{pmatrix}=0 \tag{5.7}$$

when the choice (3.3) is made for the σ-symbols.

Finally, if we express the vanishing of each element of the matrix (5.7), we immediately obtain through the relations (2.13) or (2.10) *and* (2.11) the set (5.3) of invariance conditions on the spinor components.

As a last point, we now want to show the consistency of the explicit expressions of the invariant hermitian (1, 1)-spinor and invariant four-vector when the symmetry group concerned is the group E(3). Let us take the simple case of E(3) itself: the only four-vector A invariant under E(3) is of the form $(C, 0, 0, 0)$ where C is a constant (Beckers and Hussin, to be published). In the second-rank spinor χ, as invariance under all the $\boldsymbol{P}$'s is required, all the components of χ have to be constants. Then invariance under the $\boldsymbol{J}$'s implies

$$D^{(i)}\chi=0, \qquad \forall i. \tag{5.8}$$

We then immediately obtain

$$\chi_{1\dot{2}}=\chi_{\dot{2}1}=0, \qquad \chi_{\dot{1}1}=\chi_{\dot{2}2}=C',$$

so that

$$\chi=\begin{pmatrix} C' & 0 \\ 0 & C'\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} A_0+A_3 & A_1-\mathrm{i}A_2 \\ A_1+\mathrm{i}A_2 & A_0-A_3\end{pmatrix}$$

and consequently

$$A=(C,0,0,0).$$

All the cases corresponding to the non-equivalent subgroups of E(3) will give all the results already obtained at the level of four-vectors (Beckers and Hussin, to be published).

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